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Generalized complexes and string field theory

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Abstract: I discuss the axiomatic framework of (tree-level) associative open string field theory in the presence of D-branes by considering the natural extension of the case of a single boundary sector. This leads to a formulation which is intimately connected with the mathematical theory of differential graded categories. I point out that a generic string field theory as formulated within this framework is not closed under formation of D-brane composites and as such does not allow for a unitary description of D-brane dynamics. This implies that the collection of boundary sectors of a generic string field theory with D-branes must be extended by inclusion of all possible D-brane composites. I give a precise formulation of a weak unitarity constraint and show that a minimal extension which is unitary in this sense can always be obtained by promoting the original D-brane category to an enlarged category constructed by using certain generalized complexes of D-branes. I give a detailed construction of this extension and prove its closure under formation of D-brane composites. These results amount to a completely general description of D-brane composite formation within the framework of associative string field theory.

Keywords: D-branes, Superstring Vacua.

Contents

1. Introduction

Recent work on D-brane physics has centered around the problem of describing the most general D-branes which can be introduced in a given closed string background. In the language of [28], this can formulated as follows:

Problem 1. *Given a closed string background, classify all of its open-closed extensions (i.e. describe all open-closed string theories whose closed string sector coincides with the given bulk theory).*

Here D-branes are understood in an abstract sense, namely as *boundary sectors of a string theory*. This definition does not distinguish "composite" objects such as stable non-BPS D-branes from those which can be described semiclassically through boundary conditions in a non-linear sigma model ("bootstrap").

Although not explicitly formulated as such, this problem forms the core of efforts to identify the correct classification of D-brane charges [5, 31, 32] and underlies recent work [34] aimed at understanding the physical foundations of the derived category constructions which enter the homological mirror symmetry conjecture [30].

The main difficulty of this approach comes from the observation that a consistent string theory with D-branes must include a dynamically complete set of boundary sectors. This is essentially a unitarity constraint, which can be understood as follows. Consider a string theory whose boundary sectors are described by standard D-branes, i.e. through imposing Dirichlet boundary conditions on certain submanifolds of the space-time manifold (we assume for the moment that the underlying conformal field theory admits a sigma model interpretation). It is by now well-understood that a generic[1] D-brane configuration will be unstable, a phenomenon which has been studied for parallel D-branes in both superstring field theory and bosonic string field theory (see [4] for a very partial list of references). In its standard realization, this follows from the existence of tachyon modes which condense in some non-trivial fashion. Since condensation phenomena involve off-shell string dynamics, the end result of such a process cannot be generally described in sigma model language (i.e. as a standard D-brane, defined through boundary conditions on the worldsheet fields). This basic fact implies that a unitary description of the string theory requires the inclusion of all such condensation products (which we shall call "generalized D-branes") along with the original D-branes described through boundary conditions. Indeed, a minimal requirement for unitary dynamics is that the the set of boundary sectors be closed under all physical processes.

This observation has far reaching implications for the general structure of open string field theories. In this paper, I shall focus on its consequences for *associative* tree level open string theories in the presence of D-branes. More precisely, I shall argue that systematic consideration of condensation processes leads to a description of D-brane composites through a certain type of *generalized complexes*, which (as I shall show somewhere else) turn out to be related to the twisted complexes of [12] in the particular case of topological string theories. Generalized complexes form a category which extends the D-brane category of the given theory.

Since category theory is not a traditional subject of interest among physicists, let me give a short explanation of its relevance. Categories arise in oriented open string theories in the presence of D-branes via the simple observation that one can

[1]BPS configurations in type-II theories, which were historically studied most intensively, are highly non generic (they form a set of "measure zero" in the class of all conceivable D-brane configurations).

think of D-branes as objects a and of the states of open strings stretched between them as morphisms defining string state spaces $\mathrm{Hom}(a,b)$. In this case, the natural composition of morphisms is given by the double (or basic) string product $r^{(2)}$: $\mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b) \rightarrow \mathrm{Hom}(a,c)$, which is related to the triple string correlator on the disk via:

$$\langle u, v, w \rangle = (u, r_2(v, w)) \tag{1.1}$$

for all $u \in \mathrm{Hom}(c,a), v \in \mathrm{Hom}(b,c), w \in \mathrm{Hom}(a,b)$. In this relation, $(.,.)$ is the double *conformal* correlator on the disk. In conformal field theory language, the elements of $\mathrm{Hom}(a,b)$ correspond to boundary condition changing operators if $a \neq b$, while for $a = b$ we have usual boundary operators associated with the D-brane a, which give the associated space of endomorphisms, $\mathrm{End}(a)$. To obtain a category in the standard mathematical sense, the composition of morphisms must be associative.[2] This can always be assured by going on-shell, i.e. considering the states in $\mathrm{Hom}(a,b)$ only up to BRST exact states. It is well known, however, that an on-shell analysis does not suffice to describe string theory dynamics (in the same way that it does not suffice to describe the dynamics of a gauge theory). In particular, the on-shell formalism is not appropriate when studying formation of D-brane composites. Therefore, taking BRST cohomology is not advisable as an intermediate step, and should only be done at the end of the analysis of any given problem. In physical terms, this means that we should approach non-trivial dynamical issues with the tools of string field theory. It follows that one should take $\mathrm{Hom}(a,b)$ as being *off-shell* string state spaces. In this case, one encounters a potential problem, since it is well known that the off-shell product $r^{(2)}$ need not be associative. In general, $r^{(2)}$ is only associative up to homotopy [44], i.e. up to BRST exact terms, which involve the BRST variation of the triple product. This forces one to the conclusion that a general off-shell analysis should involve a certain generalization of the classical notion of category, which was considered in [35, 36] and is known as an A_∞ category. The general mathematical theory of A_∞ categories is still in its infancy (see, however, [13]) and will be avoided in this paper. Instead, I shall restrict to the case when the product $r^{(2)}$ happens to be associative *off-shell*, the so-called case of *associative open string theory*. Well-known examples of such theories are the bosonic string field theory of [27] and the (open) holomorphic Chern-Simons theory of [37] which describes the dynamics of the open topological B-string.

To gain a general understanding of associative string field dynamics one must first enlarge the axiomatic framework of [27] to allow for the presence of D-branes. As suggested above, the correct formulation in the associative case naturally involves category theory. Since the BRST operator acts as a derivation of the string product,

[2] One must also have identity operators $Id_a \in \mathrm{Hom}(a,a)$ for all a, which are provided by the boundary vacua $|0_a\rangle$ [28].

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one is lead to consider so-called *differential graded categories*, i.e. categories whose morphism spaces $\mathrm{Hom}(a, b)$ are graded vector spaces carrying the structure of complexes, and whose differentials are compatible with morphism compositions. The relevant axioms, which are a straightforward extension of those considered in [27], are discussed in section 3 below.

It is thus clear that basic "structural" issues in D-brane physics (such as the classification of D-branes and their charges, or the extra structure which describes "unitary" brane dynamics) should be approached with category-theoretic tools. In this paper, I follow this point of view by providing an analysis of D-brane composite formation and of the unitarity constraint. This analysis is rather abstract, a price we have to pay for its extreme generality, but I hope that it successfully addresses some basic aspects of the structure of open string theory with D-branes. It also raises a set of tantalizing questions which will be briefly mentioned in the conclusions.

A complete study of the realization of our constructions in physically interesting string theories can be a daunting task, given the currently incomplete understanding of their basic algebraic and analytic aspects (not to mention serious computational difficulties). For example, very little is understood about the correct completion of the bosonic open string algebra of [27]. For the superstring, even more basic structural issues are as yet unclarified. For these reasons, a detailed analysis is currently limited to "toy models" such as the string field theories of topological strings. It is natural to expect such an analysis to make contact with the mathematical program of homological mirror symmetry [11, 18, 20, 21, 22, 23, 24, 25, 30, 33, 35, 36, 45] (see also [3, 19, 29, 40, 41, 42, 46, 47]) and we shall make some connections in a companion paper.

The organization of this paper is as follows. In section 2, we discuss the axiomatic framework of associative open string field theory in the presence of D-branes, by formulating it in the language of differential graded categories. In section 3, we recall the standard construction of the string field theory moduli space and relate it to modern deformation theory. In section 4, we consider D-brane composite formation in a string field theory with D-branes, by approaching it with the tools of section 3. We show that shifts of the string vacuum allow for a systematic description of D-brane condensation and give a precise formulation of the weak unitarity ("quasiunitarity") condition that a given open string theory is closed under D-brane formation. This translates the unitarity constraint into a condition on the underlying differential graded category. Section 4 shows how one can implement this constraint by starting with an arbitrary open string field theory with D-branes and considering a certain extension of its underlying category, which we call its "quasiunitary cover". We show that this extension can be built as a category of "generalized complexes" of objects of the original category, and prove that it satisfies the quasiunitarity constraint. Section 5 suggests some directions for further research. Appendix A collects some basic concepts relevant to differential graded categories.

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2. The axiomatic framework of associative open string field theory with D-branes

It can be shown that the a general tree level open string field theory in the presence of D-branes can be described through an A_∞ category [35] (this follows from a relatively straightforward, but rather tenuous, extension of the axiomatic analysis of [43, 44]). In this paper I shall restrict to the associative case, which can be described in a more standard mathematical language as follows:[3]

A (tree level) open string field theory in the presence of D-branes is given by the following data:

1. A differential graded $\mathbf{C}$-linear category $\mathcal{A}$

2. For each pair of objects a, b of $\mathcal{A}$, an invariant non-degenerate bilinear and graded-symmetric form ${}_{ab}(.,.)_{ba} : \mathrm{Hom}(a,b) \times \mathrm{Hom}(b,a) \to \mathbf{C}$ of degree 3.

Let us explain each piece of data. By a $\mathbf{C}$-*linear category* we mean a category whose morphism spaces are complex vector spaces and whose morphism compositions are bilinear maps. A *graded linear category* is a linear category whose morphism spaces are $\mathbf{Z}$-graded, i.e. $\mathrm{Hom}(a,b) = \oplus_{k\in\mathbb{Z}}\mathrm{Hom}^k(a,b)$, and such that morphism compositions are homogeneous of degree zero, i.e.

$$|uv| = |u| + |v| \,, \qquad \text{for all homogeneous } u \in \mathrm{Hom}(b,c)\,, \qquad v \in \mathrm{Hom}(a,b)\,, \tag{2.1}$$

where $|.|$ denotes the degree of a homogeneous element. In a *differential graded linear category* (dG category, for short), the morphism spaces $\mathrm{Hom}(a,b)$ are further endowed with nilpotent operators Q_{ab} (i.e. $Q_{ab}^2 = 0$) of degree $+1$ which act as derivations of morphism compositions:

$$Q_{ac}(uv) = Q_{bc}(u)v + (-1)^{|u|}uQ_{ab}v\,, \qquad \text{for homogeneous } u \in \mathrm{Hom}(b,c)\,,\; v \in \mathrm{Hom}(a,b)\,. \tag{2.2}$$

This can also be expressed as the requirement that the composition maps $\mathrm{Hom}(b,c)\otimes \mathrm{Hom}(a,b) \to \mathrm{Hom}(a,c)$ are morphisms of complexes, where $\mathrm{Hom}(b,c)\otimes\mathrm{Hom}(a,b)$ carries the structure of total complex induced from its components, i.e.

$$\begin{aligned}[\mathrm{Hom}(b,c)\otimes\mathrm{Hom}(a,c)]^k &= \oplus_{i+j=k}\mathrm{Hom}^i(b,c)\otimes\mathrm{Hom}^j(a,b)\,,\\ Q(u\otimes v) &= Q_{bc}u\otimes v + (-1)^{|u|}u\otimes Q_{ab}v\,.\end{aligned} \tag{2.3}$$

[3] Note that we are using conventions which are slightly different from those of [44], but better adapted to making contact with the physics literature. For the bosonic string, our grading $|.|$ is given by the ghost number gh, while the grading of [44] is given by $1 - gh$. Our conventions also differ from those of [27].

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Graded symmetry of the bilinear forms means:

$$ {}_{ab}(u,v)_{ba} = (-1)^{|v||u|}_{ba}(v,u)_{ab} \tag{2.4}$$

for homogeneous elements $u \in \mathrm{Hom}(a,b)$ and $v \in \mathrm{Hom}(b,a)$. The degree 3 constraint is:

$$ {}_{ab}(u,v)_{ba} = 0\,, \qquad \text{unless } |u|+|v| = 3\,. \tag{2.5}$$

The bilinear form is required to be invariant with respect to the differentials Q_{ab} and with respect to morphism compositions:

$$ {}_{ab}(Q_{ab}(u),v)_{ba} + (-1)^{|u|}(u, Q_{ba}(v)) = 0\,, \qquad \text{for } u \in \mathrm{Hom}(a,b)\,, \quad v \in \mathrm{Hom}(b,a) \tag{2.6}$$

and

$$ {}_{ca}(u,vw)_{ac} = {}_{ba}(uv,w)_{ab}\,, \qquad \text{for } u \in \mathrm{Hom}(c,a)\,, \quad v \in \mathrm{Hom}(b,c)\,, \quad w \in \mathrm{Hom}(a,b)\,. \tag{2.7}$$

In physical theories such as the bosonic string, one also has antilinear involutions obeying certain constraints involving the bilinear forms and the BRST operators. This extra structure need not be present in a topological field theory[4] and will not be considered in this paper. It is not hard to extend the discussion below by including such conjugations, but this adds little to our results while cluttering the presentation.

The physical interpretation of this data is as follows. The objects a are identified as D-branes (in an abstract sense), while $\mathrm{Hom}(a,b)$ is the (off-shell) state space of strings stretching from a to b. These will be called the *boundary sectors* of the theory. The sectors $\mathrm{Hom}(a,a) = \mathrm{End}(a)$ are the diagonal boundary sectors, while $\mathrm{Hom}(a,b)$ with $a \neq b$ are the off-diagonal, or boundary condition changing sectors. The nilpotent operators Q_{ab} give the BRST charge in the sectors $\mathrm{Hom}(a,b)$, while the $\mathbf{Z}$-grading of $\mathrm{Hom}(a,b)$ is induced by a grading on the boundary worldsheet fields. In a physical theory such as the open bosonic string, this is simply the ghost number degree, while for A/B topological strings it is the grading associated with the anomalous $\mathrm{U}(1)$ symmetry of the topological ("twisted") $N = 2$ superconformal algebra.

The non-degenerate bilinear forms ${}_{ab}(.,.)_{ba}$ induce linear isomorphisms:

$$ bpz_{ab} : \mathrm{Hom}(a,b) \xrightarrow{\approx} \mathrm{Hom}(b,a)^* \tag{2.8}$$

through the standard prescription:

$$ bpz_{ab}(u)(v) :=_{ba} (v,u)_{ab}\,, \qquad \text{for all } u \in \mathrm{Hom}(a,b)\,, \quad v \in \mathrm{Hom}(b,a)\,. \tag{2.9}$$

These are the usual BPZ conjugations.

[4]For example, it is irrelevant for the B-model [37, 38, 39], whose string field action is allowed to be complex.

2.1 On-shell formalism

One can recover the on-shell description by taking the BRST cohomology of all boundary state spaces $\mathrm{Hom}(a, b)$. It is easy to see that this produces a graded category, the so-called *cohomology category* of the dG category $\mathcal{A}$ (see appendix A). This has the same objects as $\mathcal{A}$ but morphism spaces given by the BRST cohomology:

$$\mathrm{Hom}_{H(\mathcal{A})}(a, b) = H^*_Q(\mathrm{Hom}_{\mathcal{A}}(a, b)) . \tag{2.10}$$

The morphism compositions of $H(\mathcal{A})$ are induced from those of $\mathcal{A}$ and are well defined by virtue of condition (2.2). Moreover, condition (2.6) implies that the bilinear forms of $\mathcal{A}$ descend to forms ${}_{a,b}(.,.)_{ba}$ on $\mathrm{Hom}_{H(\mathcal{A})}(a, b) \otimes \mathrm{Hom}_{H(\mathcal{A})}(b, a)$, which are invariant with respect to composition of morphisms in $H(\mathcal{A})$. This recovers the boundary part of the data discussed in [28].

2.2 The total boundary state space

The structure above can be related to the more familiar data of [27] (see [26] for a review) by defining the total boundary state space $\mathcal{H} = \oplus_{a,b}\mathrm{Hom}(a, b)$. Then one defines the total BRST charge via:

$$Q = \oplus_{a,b} Q_{ab} , \tag{2.11}$$

and the total boundary composition $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$:

$$uv = \oplus_{ac} \left[\sum_b u_{bc} v_{ab} \right] , \tag{2.12}$$

for $u = \oplus_{ab} u_{ab}$, $v = \oplus_{ab} v_{ab}$ with $u_{ab}, v_{ab} \in \mathrm{Hom}(a, b)$. Moreover, the bilinear forms ${}_{ab}(.,.)_{ba}$ induce the total boundary form:

$$(u, v) = \sum_{ab} {}_{ab}(u_{ab}, v_{ba})_{ba} . \tag{2.13}$$

It is easy to see that the resulting "total" data satisfies the axioms of [37], i.e.

(I) $\mathcal{H}$ together with Q and the total boundary composition is a differential graded associative algebra

(II) The total boundary bilinear form $(.,.)$ is non degenerate and invariant with respect to the boundary BRST operator and the boundary total product.

However, our theory contains extra structure, encoded in the category-theoretic properties of Q, $(.,.)$ and the boundary product. This structure amounts to the statement that the total boundary data decompose in the form given above. For example, the boundary product must be non zero only on subspaces of the form

$\mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b)$, which it must take into the subspace $\mathrm{Hom}(a,c)$ of $\mathcal{H}$. Also note that, in practice, the number of objects (D-branes) will typically be infinite (infinite of the power of the continuum, which makes the interpretation in terms of the total boundary state space somewhat tenuous).

The string field action on the total boundary space has the form considered in [37] (we assume that the given background satisfies the string equations of motion, so that the action does not contain a linear term):

$$S(\phi) = \frac{1}{2}(\phi, Q\phi) + \frac{1}{3}(\phi, \phi\phi)\,, \tag{2.14}$$

where the string field $\phi = \oplus_{a,b}\phi_{ab}$ (with $\phi_{ab} \in \mathrm{Hom}(a,b)$) is constrained to be of degree one:[5]

$$|\phi_{ab}| = 1\,. \tag{2.15}$$

The string field action can be trivially expanded in boundary sectors:

$$S(\phi) = \frac{1}{2}\sum_{ab} {}_{ab}(\phi_{ab}, \phi_{ba})_{ba} + \frac{1}{3}\sum_{abc} {}_{ca}(\phi_{ca}, \phi_{bc}\phi_{ac})_{ac}\,. \tag{2.16}$$

In a realistic case (backgrounds containing an infinity of D-branes), this is of course a formal expression, unless one assumes that only a finite number of boundary sectors can be simultaneously excited (in this case, the string field is restricted to have non-zero value only in a finite number of boundary sectors).

3. The string field theory moduli space

In this section we recall the basic procedure for building the string field theory moduli space and discuss its connection with shifts of the string vacuum. All of the results presented below are well known or implicit in the literature, and my only contribution is to reformulate some of them in the language of deformation theory.

Let us consider a string field theory based on a dG algebra $\mathcal{H}$, which can be the boundary state space of a theory with one D-brane or the total boundary space of a theory with multiple D-branes. In this section, we work at the level of the algebra $\mathcal{H}$, temporarily ignoring the category structure.

Given such a theory, a shift of the background (i.e. of the associated string vacuum) corresponds to a translation $\phi \to \phi + q$ of the string field, which induces a translation $S(\phi) \to S'(\phi) = S(\phi + q)$ of the string field action.[6] Since the string field has degree one, we must assume $|q| = 1$. The shifted action can be expanded as:

$$S'(\phi) = S(\phi + q) = S(q) + (r_0, \phi) + \tilde{S}(\phi)\,, \tag{3.1}$$

[5]In a theory admitting antilinear involutions, one also imposes a reality constraint on ϕ.
[6]That is, we set $\phi = \phi' + q$ so that $S(\phi) = S(\phi' + q) = S'(\phi')$, then we rename ϕ' as ϕ.

where:

$$r_0 = Qq + q^2\,,$$
$$\tilde{S}(\phi) = \frac{1}{2}(\phi, (Q+2q)\phi) + \frac{1}{3}(\phi, \phi\phi)\,. \tag{3.2}$$

One can bring $\tilde{S}$ to the form (2.14), by defining a shifted BRST operator:

$$Q'u = Qu + [q, u]\,, \qquad \text{(for any } u \in \mathcal{H})\,, \tag{3.3}$$

where $[,]$ is the graded commutator on $\mathcal{H}$ induced by the associative product:

$$[u, v] = uv - (-1)^{|u||v|}vu\,. \tag{3.4}$$

Then $Q'\phi = (Q + 2q)\phi$ (since $|q| = |\phi| = 1$) and we can write:

$$\tilde{S}(\phi) = \frac{1}{2}(\phi, Q'\phi) + \frac{1}{3}(\phi, \phi\phi)\,. \tag{3.5}$$

The linear term in (3.1) signals the presence of string tadpoles in an expansion around the new vacuum. The condition that this vacuum satisfies the string equations of motion (i.e. the tadpole cancellation constraint) takes the form $r_0 = 0$, i.e.

$$Qq + qq = 0\,. \tag{3.6}$$

On the other hand, it is easy to compute:

$$(Q')^2u = [Qq + q^2, u]\,, \tag{3.7}$$

so that the operator Q' is nilpotent on $\mathcal{H}$ precisely when the tadpole cancellation condition (3.6) holds. In this case, Q' is obviously a degree one derivation of the associative product, so that $(\mathcal{H}, Q', \cdot)$ is a dG algebra. Since the constant term $S(q)$ is irrelevant, it follows that a shift of the vacuum is equivalent with the replacement $S(\phi) \to \tilde{S}(\phi)$, which in turn amounts to the modification $Q \to Q'$ of the BRST charge. Moreover, the tadpole cancellation constraint (3.6) is equivalent with the condition that the deformed structure $(\mathcal{H}, Q', \cdot)$ remains a dG algebra.

One can recognize the standard ingredients of deformation theory [8] (see also [7]). Indeed, since $|q| = 1$, we have $[q, q] = 2q^2$ and we can re-write the basic constraint (3.6) in the standard form:

$$Qq + \frac{1}{2}[q, q] = 0\,. \tag{3.8}$$

This is the Maurer-Cartan equation in the dG Lie algebra $(\mathcal{H}, d, [.,.])$ induced by our differential graded associative algebra. Moreover, two infinitesimal shifts q, q' should be identified if they are related through:

$$q' = q + \beta\,, \tag{3.9}$$

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where α satisfies:

$$Q\alpha + [q, \alpha] = \beta \,. \tag{3.10}$$

It follows the moduli space of string vacua is the moduli space associated with the deformation problem described by the Maurer-Cartan equation (3.8). Its tangent space at the origin (i.e. at the point corresponding to the original vacuum) is given by the linearization of (3.8) and the zeroth order approximation to (3.10), which define the first degree Q-cohomology $H^1_Q(\mathcal{H})$.

4. Shifts of a string vacuum with D-branes and formation of D-brane composites

In the case of multiple D-branes, one can apply the general discussion of the previous subsection to the dG associative algebra $\mathcal{H} = \oplus_{a,b}\,\mathrm{Hom}(a, b)$. In this case, a deformation q has the expansion $q = \sum_{a,b} q_{ab}$, with $q_{ab} \in \mathrm{Hom}^1(a, b)$. Expanding the equations above into boundary sectors leads to the tadpole cancellation constraint:

$$Q_{ab}q_{ab} + \sum_c q_{cb}q_{ac} = 0 \,, \tag{4.1}$$

and to the following expression for the shifted BRST charge:

$$Q'u = \oplus_{a,b} \left[Q_{ab}u + \sum_c \left(q_{cb}u_{ac} - (-1)^{|u_{cb}|} u_{cb}q_{ac} \right) \right] , \tag{4.2}$$

where $u = \oplus_{ab}u_{ab}$, with $u_{ab} \in \mathrm{Hom}(a, b)$. The degree one condition on q_{ab} assures that q is a homogeneous element of degree one, and thus Q' is a derivation of degree $+1$. It also assures that q can be thought of as a shift of the degree one string field ϕ.

4.1 D-brane composites and pseudocomplexes

Let us define a *pseudocomplex* to be given by a set S of objects of $\mathcal{A}$ together with a set of degree one morphisms $q_{ab} \in \mathrm{Hom}^1(a, b)$ $(a, b \in S)$ which satisfy the tadpole cancellation constraint (4.1). More precisely, this condition has to be satisfied for *all* objects a, b in $\mathcal{A}$, where we define $q_{de} = 0$ if d or e does not belong to the set S. An example of pseudocomplex is shown in figure 1.

In view of the above, pseudocomplexes describe vacuum shifts of our string theory, and allow one to build its vacuum moduli space by solving the moduli problem defined by the Maurer-Cartan equation (4.1). The fact that we have multiple boundary sectors implies that a deformation through such a complex corresponds to condensation of boundary states q_{ab}, leading to a D-brane composite. Indeed, the standard interpretation of a shift by q is that the components q_{ab} of the string field acquire VEV's, which is the original physical motivation for shifting the string vacuum. This suggests that *generalized complexes describe D-brane composites.* To justify this more formally, notice that the shifted theory does not preserve the origi-

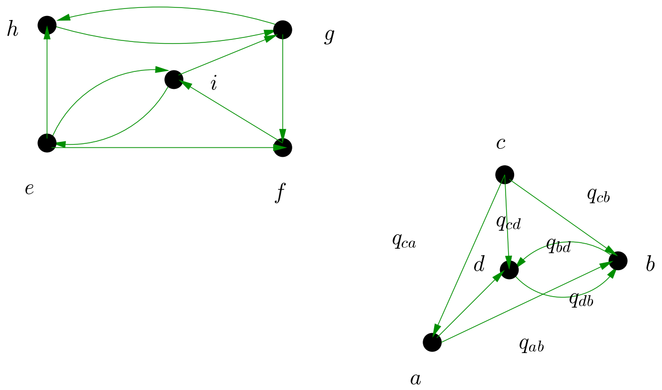

Figure 1: A pseudocomplex with two connected components. For the second component, we also indicate the labeling of morphisms. Note that the objects $a..i$ sitting at the nodes are all distinct.

nal decomposition $\mathcal{H} = \oplus_{a,b}\mathcal{H}_{ab}$ into boundary sectors, in the sense that the shifted BRST operator Q' does not map every subspace $\mathcal{H}_{ab}$ into $\mathcal{H}_{ab}$, thus violating our axioms. This can be remedied by taking the approach of [28], namely we view the data $(\mathcal{H}, Q', \cdot)$ as fundamental and look for a new decomposition of $\mathcal{H}$ into boundary sectors such that the axioms of section 2 are satisfied.

To make this precise, notice that a pseudocomplex can be viewed as a graph Γ whose nodes are the objects $a \in S$ and whose arrows are given by those morphisms q_{ab} which are *non vanishing* (visualized as an arrow from a to b). We say that the complex q is *irreducible* (or *connected*) if this graph is connected, i.e. any two distinct objects $a, b \in S$ can be connected through a sequence of arrows belonging to the graph.

4.1.1 Shifts by irreducible q

For an irreducible q, the total boundary space $\mathcal{H}$ admits the decomposition:

$$\begin{aligned}\mathcal{H} = {} & [\oplus_{a,b\in S}\mathrm{Hom}(a,b)] \oplus [\oplus_{a,b\notin S}\mathrm{Hom}(a,b)] \oplus (\oplus_{b\notin S}[\oplus_{a\in S}\mathrm{Hom}(a,b)]) \oplus \\ & \oplus (\oplus_{a\notin S}[\oplus_{b\in S}\mathrm{Hom}(a,b)])\,,\end{aligned}$$

which preserves all axioms if the terms in square brackets are viewed as the new boundary sectors. This amounts to viewing the shifted vacuum as the category $\mathcal{A}_q$ obtained from $\mathcal{A}$ by collapsing the set of objects S to a single new object $*$. Beyond $*$, this category contains all objects of $\mathcal{A}$ which do not belong to S, and its morphism spaces are given by:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}_q}(*,*) &= \oplus_{a,b\in S}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(a,*) &= \oplus_{b\in S}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } a \text{ not in } S\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(*,b) &= \oplus_{a\in S}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } b \text{ not in } S\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(a,b) &= \mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } a,b \text{ not in } S\,, \end{aligned} \tag{4.3}$$

and with the morphism compositions induced from $\mathcal{H}$. It is easy to see that Q' preserves these subspaces of $\mathcal{H}$. We define the BRST operators on the new boundary sectors to be given by the restriction of Q', while the bilinear forms are given by the restriction of $(.,.)$. For example, the BRST operator Q_{**} acting on $\mathrm{Hom}_{\mathcal{A}_q}(*,*)$ is given by:

$$(Q'_{**}u) = \oplus_{a,b\in S}\left[Q_{ab}u_{ab} + \sum_{c\in S}\left(q_{cb}u_{ac} - (-1)^{|u_{cb}|}u_{cb}q_{ac}\right)\right], \tag{4.4}$$

for $u = \oplus_{a,b\in S}u_{ab} \in \mathrm{Hom}(*,*)$, with $u_{ab} \in \mathrm{Hom}(a,b)$. Its cohomology $H^*_{Q'_{**}}(\mathrm{Hom}_{\mathcal{A}_q}(*,*))$ is the space of on-shell states of open strings in the new diagonal boundary sector produced by condensation of the operators q_{ab}. It is easy to see that $\mathcal{A}_q$ endowed with this structure satisfies Axioms (1) and (2) of section 2. This gives the description of our string field theory after shifting the vacuum. The new decomposition into boundary sectors shows that the pseudocomplex q can be identified with the new object $*$, the D-brane composite obtained by condensing the boundary operators $(q_{ab})_{a,b\in S}$.

4.1.2 Shifts by reducible q

If the complex is not connected, then each of the connected components Λ of the associated graph Γ defines a sub-collection $q_\Lambda = (q_{cd})_{c,d\in S_\Lambda}$, where S_Λ is the set of objects of $\mathcal{A}$ associated with its vertices. In this case, the sets S_Λ form a partition (disjoint union decomposition) of the set S, and it is easy to see that each collection q_Λ is itself a pseudocomplex, i.e. it satisfies the tadpole cancellation constraint (4.1). Hence the reducible complex q can be viewed as the collection of irreducible complexes $(q_\Lambda)_{\Lambda\in\mathcal{P}}$, where $\mathcal{P}$ is the set of connected components of Γ.

In this case, an analysis similar to that above leads to the conclusion that q gives a collection of irreducible D-branes described by the connected complexes q_Λ. Each composite D-brane arises through condensation of the boundary operators in q_Λ, and these condensation processes are independent since there is no non-zero boundary operator which condenses between D-branes belonging to distinct subcomplexes q_{Λ_1} and q_{Λ_2}. The associated category $\mathcal{A}_q$ is obtained by collapsing each of the sets S_Λ to a different object $*_\Lambda$ (figure 2).

The formal definition of $\mathcal{A}_q$ is as follows. The objects of $\mathcal{A}_q$ are those objects of $\mathcal{A}$ which do not belong to S, together with the new objects $*_\Lambda$ ($\Lambda \in \mathcal{P}$). Its morphism spaces are given by:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Sigma) &= \oplus_{a\in S_\Lambda, b\in S_\Sigma}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, b) &= \oplus_{a\in S_\Lambda}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } b\notin S\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(a, *_\Sigma) &= \oplus_{b\in S_\Sigma}\mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } a\notin S\,, \\ \mathrm{Hom}_{\mathcal{A}_q}(a,b) &= \mathrm{Hom}_{\mathcal{A}}(a,b)\,, \qquad \text{for } a,b\notin S\,, \end{aligned} \tag{4.5}$$

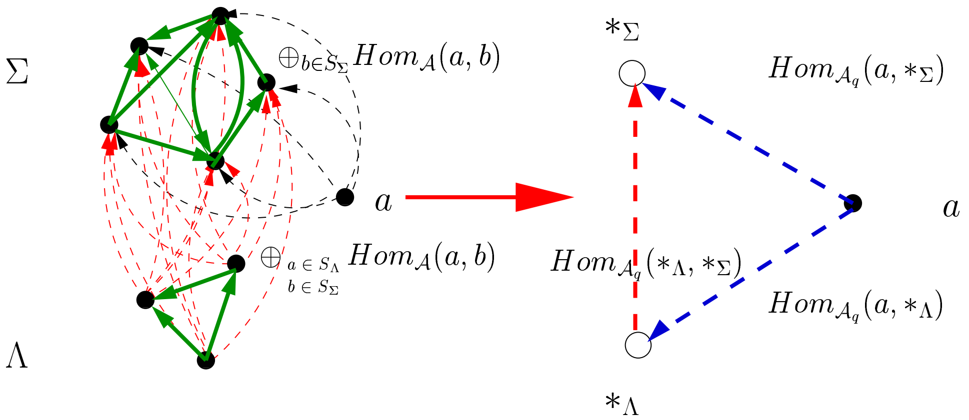

Figure 2: Formation of two irreducible D-brane composites. After performing the vacuum shift, each of the two connected subgraphs on the left (the arrows of which represent a morphism of the associated complex q) is replaced by a single object (the two unfilled circles on the right). We also draw one of the objects which does not belong to the associated complex, and show how the various morphism spaces involving D-brane composites are built from the original morphisms (the dashed lines). We do not indicate the morphisms making up $\mathrm{Hom}(a, *_\Sigma)$, in order to prevent excessive clutter of the figure.

with the grading induced from that on $\mathrm{Hom}(a,b)$. Morphism compositions are given by restriction of the associative composition in $\mathcal{H}$. The BRST operator (4.2) induces the BRST charges in these new boundary sectors:

$$\begin{aligned}
Q'_{*_\Lambda,*_\Sigma}u &= \oplus_{a\in S_\Lambda, b\in S_\Sigma}\left[Q_{ab}u_{ab} + \sum_{c\in S_\Sigma} q_{cb}u_{ac} - \sum_{c\in S_\Lambda}(-1)^{|u_{cb}|}u_{cb}q_{ac}\right],\\
Q'_{a,*_\Sigma}u &= \oplus_{b\in S_\Sigma}\left[Q_{ab}u_{ab} + \sum_{c\in S_\Sigma} q_{cb}u_{ac}\right](a\not\in S)\,,\\
Q'_{*_\Lambda,b}u &= \oplus_{a\in S_\Lambda}\left[Q_{ab}u_{ab} - \sum_{c\in S_\Lambda}(-1)^{|u_{cb}|}u_{cb}q_{ac}\right](b\not\in S)\,,\\
Q'_{ab} &= Q'_{ab}(a,b\not\in S)\,.
\end{aligned} \tag{4.6}$$

We shall call $\mathcal{A}_q$ the *contraction* of the category $\mathcal{A}$ along q. The cohomology of $Q'_{*_\Lambda,*_\Sigma}$ on the space $\mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda,*_\Sigma)$ gives the on-shell state space of strings stretching between the composites $*_\Lambda$ and $*_\Sigma$. As above, we have induced bilinear forms (from restriction of $(.,.)$) and it is easy to see that $\mathcal{A}_q$ satisfies Axioms (1) and (2) of section 2. We conclude that *the shifted string field theory is described by the contracted category* $\mathcal{A}_q$ (figure 3).

For reader's convenience, I list below the explicit morphism composition rules and bilinear forms on the category $\mathcal{A}_q$.

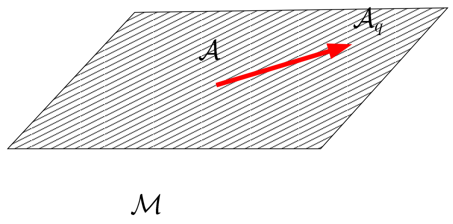

Figure 3: Shifting the vacuum changes the description of our string field theory. The fact that the category structure changes signals the formation of D-brane composites.

1. Morphism compositions. For $u \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Sigma, *_\Theta)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Sigma)$:

$$uv = \oplus_{a\in S_\Lambda, b\in S_\Theta}\left[\sum_{c\in S_\Sigma} u_{cb}v_{ac}\right] \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Theta)\,. \tag{4.7}$$

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Sigma, c)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Sigma)$ $(c \notin S)$:

$$uv = \sum_{a\in S_\Lambda, b\in S_\Sigma} u_b v_{ab} \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, c)\,, \tag{4.8}$$

where $u = \sum_{b\in S_\Sigma} u_b$ with $u_b \in \mathrm{Hom}_{\mathcal{A}}(b, c)$.

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Sigma)$, $v \in \mathrm{Hom}_{\mathcal{A}_q}(a, *_\Lambda)$ $(a \notin S)$:

$$uv = \oplus_{c\in S_\Sigma}\left[\sum_{b\in S_\Lambda} u_{bc}v_b\right] \in \mathrm{Hom}_{\mathcal{A}_q}(a, *_\Sigma)\,, \tag{4.9}$$

where $v = \oplus_{b\in S_\Lambda} v_b$ with $v_b \in \mathrm{Hom}_{\mathcal{A}}(a, b)$.

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(b, c) = \mathrm{Hom}_{\mathcal{A}}(b, c)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(a, b) = \mathrm{Hom}_{\mathcal{A}}(a, b)$, with $a, b \notin S$b, the composition uv is given by the morphism composition of $\mathcal{A}$.

2. Bilinear forms. For $u \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Sigma, *_\Lambda)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, *_\Sigma)$:

$$(u, v) = \sum_{a\in S_\Sigma, b\in S_\Lambda} (u_{ab}, v_{ba})\,. \tag{4.10}$$

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, b)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(b, *_\Lambda)$ $(b \notin S)$:

$$uv = \sum_{a\in S_\Lambda} (u_a, v_a)\,, \tag{4.11}$$

where $u = \sum_{a\in S_\Lambda} u_a$ with $u_a \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ and $v = \oplus_{a\in S_\Lambda} u_a$ with $u_a \in \mathrm{Hom}_{\mathcal{A}}(b, a)$.

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(a, *_\Lambda)$, $v \in \mathrm{Hom}_{\mathcal{A}_q}(*_\Lambda, a)$ $(a \notin S)$:

$$uv = \sum_{b\in S_\Lambda} (u_b, v_b)\,. \tag{4.12}$$

where $u = \oplus_{b\in S_\Lambda} u_b$ with $u_b \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ and $v = \sum_{b\in S_\Lambda} v_b$ with $v_b \in \mathrm{Hom}_{\mathcal{A}}(b, a)$.

For $u \in \mathrm{Hom}_{\mathcal{A}_q}(a,b) = \mathrm{Hom}_{\mathcal{A}}(a,b)$ and $v \in \mathrm{Hom}_{\mathcal{A}_q}(b,a) = \mathrm{Hom}_{\mathcal{A}}(b,a)$ with $a, b \notin S$, (u,v) is given by the bilinear form of $\mathcal{A}$.

All of the sums involved in these expressions are well defined if one restricts to pseudocomplexes whose underlying set of objects S is finite.

4.2 The quasiunitarity constraint

We are now ready to formulate precisely the unitarity constraint discussed in the introduction.

Definition 2. *We say that a string field theory based on a dG category is* quasiunitary *(or* closed under formation of D-brane composites*) if for any pseudocomplex q, the contracted category $\mathcal{A}_q$ is dG-equivalent with a full subcategory*[7] *of the category $\mathcal{A}$ via a functor which preserves the bilinear forms.*

Two dG categories are dG-equivalent if they are related by an equivalence of categories which preserves the differentials on morphisms. A formal definition can be found in appendix A.

This definition formalizes the intuition that performing a shift of the vacuum should not produce new boundary sectors. A "unitary" theory should be such that formation of D-brane composites always "reduces" the set of objects (D-brane) originally available. This encodes the basic constraint that the original theory is "complete", i.e. the state space is large enough in order to describe the dynamics.

It is clear that a "generic" string field theory in the sense of section 2 does not satisfy this constraint. For examples, topological B-type string theory on a Calabi-Yau manifold, in the presence of D-branes described by holomorphic vector bundles satisfies Axioms (1) and (2) of section 2 but it is not unitary in this sense, since a shift of the vacuum leads to D-brane composites described by complexes of holomorphic vector bundles (and, in fact, to much more general objects, as we shall show in a companion paper). Similar remarks apply to physical string theories, such as superstring field theory on a Calabi-Yau manifold in the presence of D-branes. A theory which is not unitary at least in this weak sense is physically incomplete if viewed as a space-time description of D-brane dynamics, though it is of course well defined as an abstract object. We propose that any open string field theory with D-branes should be formulated in a quasi-unitary manner.

5. The quasi-unitary cover of an open string theory

Given an open string field theory based on the category $\mathcal{A}$, it seems intuitively plausible that one could construct a quasi-unitary completion of that theory by

[7] Remember that a full subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is a subcategory such that $\mathrm{Hom}_{\mathcal{B}}(a,b) = \mathrm{Hom}_{\mathcal{A}}(a,b)$ for any two objects $a, b \in \mathcal{B}$.

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enlarging the class of objects through inclusion of all possible D-brane composites (= pseudocomplexes). It turns out that it is slightly subtle to give a precise formulation of this intuition, due to the following observation (*the problem of infinite recursion*).

The reason for expecting a quasiunitary completion is that one can consider all possible D-brane composites and enlarge the original category $\mathcal{A}$ by adding them as new objects. It is in fact possible to formulate a category $p(\mathcal{A})$ of all pseudocomplexes over $\mathcal{A}$ by simply using relations (4.5) and (4.6) to define the morphism spaces between pseudocomplexes and the associated BRST operators, and composing morphisms through the obvious analogue of the composition in $\mathcal{A}_q$.[8] This category contains $\mathcal{A}$ as a full subcategory upon identifying objects of $\mathcal{A}$ with the associated pseudocomplex based on one object carrying the zero morphism. However, it turns out that the category $p(\mathcal{A})$ is *not* closed under formation of D-brane composites. It follows that one has to repeat the process by building the category $p^2(\mathcal{A}) = p(p(\mathcal{A}))$ of pseudocomplexes over $p(\mathcal{A})$ (which corresponds to including composites arising by condensing boundary operators between various D-brane composites), then the categories $p^3(\mathcal{A})$, $p^4(\mathcal{A})$ and so forth. While this approach is certainly allowed, a quasiunitary completion would only be obtained as a limit of $p^n(\mathcal{A})$ when n tends to infinity, and its explicit description seems difficult to extract. Moreover, one would have to define in what sense one takes such a limit (presumably an inductive limit).

Below, I show that this problem can be avoided by the deceivingly simple device of allowing for complexes based on *sequences* of objects of $\mathcal{A}$. This amounts to passing from pseudocomplexes to so-called *generalized complexes*, which are essentially pseudocomplexes whose underlying objects are allowed to be identical (since they are based on a sequence of objects rather than a set of objects, and a sequence may have repetitions). This apparently trivial extension suffices to "sum up" all of the recursive extensions $p^n(\mathcal{A})$, as one can prove directly that the resulting category of generalized complexes is quasiunitary. Note, however, that a generalized complex does not have a very direct interpretation as a D-brane composite, unless one agrees that it makes sense to condense multiple copies of a D-brane with itself.

Before proceeding with a more detailed explanation of this construction, let me comment on the relation between our generalized complexes and apparently similar complexes of D-branes considered in the work of [15, 34] (see also [16] for related issues). In the approach of those papers, one looks at type-IIB superstring theory on Calabi-Yau manifolds and obtains complexes based on sequences of holomorphic vector bundles. This allows for repetitions of the same bundle in a sequence, but such sequences do *not* correspond to a generalized complex of D-branes in the sense of our paper. The reason in that, in the approach of [15, 34], two consecutive copies of the same bundle in a sequence must correspond to a brane-antibrane pair, so that the

[8]Since it certainly possible to perform *distinct* shifts of the vacuum leading to D-brane composites based on non-disjoint collections of objects of $\mathcal{A}$ (i.e. the associated pseudocomplexes share some of their objects), this involves a slight generalization.

associated D-branes are physically distinct even though the underlying holomorphic vector bundles are identical. In a generalized complex as considered in this paper, one has multiple copies of the same *physical* D-brane, identified with an object of $\mathcal{A}$. As will be shown somewhere else, applying our construction to the open topological B-model gives (among much more general objects) sequences of holomorphic vector bundles (with repetitions allowed) with morphisms given by bundle-valued differential forms of arbitrary degree. This suffices to recover the complexes of degree zero morphisms considered in [15, 34] and much more, but via a rather different procedure, which consists of considering the so-called shift completion of the category of holomorphic vector bundles. Moreover, note that our formalism applies perfectly well to associative bosonic string field theory, for which no simple analogue of the arguments of [15, 34] seems to exist.[9]

5.1 Motivation

The proposal below is based on our discussion of formation of D-brane composites. As mentioned above, a unitary description requires passage from pseudocomplexes to a more general construction ("generalized complexes") which involve *sequences* of objects of $\mathcal{A}$. Before presenting the construction itself, let me outline the motivation for introducing such objects. Consider minimally extending the category $\mathcal{A}$ to a dG category $\mathcal{B}$ which satisfies the quasiunitarity constraint. It is clear that a minimal extension must amount at least to addition of all pseudocomplexes (= D-brane composites). Hence the enlarged category $\mathcal{B}$ contains at least all pseudocomplexes based on the objects of $\mathcal{A}$. Note that an object a of $\mathcal{A}$ can be identified with the one-morphism pseudocomplex (0_a), where 0_a is the zero morphism in $\mathrm{Hom}_{\mathcal{A}}(a, a)$.

Now consider an object a of $\mathcal{A}$ and a pseudocomplex $q = (q_{aa})$ over $\mathcal{A}$ with a single non-zero morphism $q_{aa} \in \mathrm{Hom}^1(a, a)$ based on a. The tadpole cancellation constraint for such a complex reads:

$$Q_{aa}q_{aa} + q_{aa}q_{aa} = 0\,, \tag{5.1}$$

which is the Maurer-Cartan equation for vacuum deformations in the boundary sector $\mathrm{Hom}_{\mathcal{A}}(a, a)$ (it follows that such a complex describes deformations of the D-brane a of $\mathcal{A}$). We shall make the assumption that the associated morphism spaces and BRST differentials in $\mathcal{B}$ are given by the generalization of (4.5) and (4.6) to the case of pseudocomplexes based on non-disjoint sets of objects (we identify a with the

[9] The reader has been warned that we are talking about something quite different from the brane-antibrane sequences of [15, 34], in spite of what a shallow analogy might suggest. To the extent that open superstring field theory satisfies our axioms (which is not entirely clear), the original category $\mathcal{A}$ would contain both branes and antibranes, which in our treatment are distinct objects. We are interested in sequences with repetitions of the same D-brane, and not in sequences of brane/antibrane pairs.

pseudocomplex (0_a)):[10]

$$\mathrm{Hom}_{\mathcal{B}}(a,a)=\mathrm{Hom}_{\mathcal{B}}(a,q)=\mathrm{Hom}_{\mathcal{B}}(q,a)=\mathrm{Hom}_{\mathcal{B}}(q,q)=\mathrm{Hom}_{\mathcal{A}}(a,a) \tag{5.2}$$

and:

$$\begin{aligned}
Q^{\mathcal{B}}_{aa}u &= Q_{aa}u\,, \qquad \text{for } u\in \mathrm{Hom}_{\mathcal{B}}(a,a)\,,\\
Q^{\mathcal{B}}_{aq}u &= Q_{aa}u+q_{aa}u\,, \qquad \text{for } u\in \mathrm{Hom}_{\mathcal{B}}(a,q)\,,\\
Q^{\mathcal{B}}_{qa}u &= Q_{qa}u-(-1)^{|u|}uq_{aa}\,, \qquad \text{for } u\in \mathrm{Hom}_{\mathcal{B}}(q,a)\,,\\
Q^{\mathcal{B}}_{qq}u &= Q_{aa}u+q_{aa}u-(-1)^{|u|}uq_{aa}\,, \qquad \text{for } u\in \mathrm{Hom}_{\mathcal{B}}(q,q)\,.
\end{aligned} \tag{5.3}$$

This assumption is natural in view our previous discussion of D-brane condensation.

Both the complex q and the D-brane $a\equiv(0_a)$ are objects of the enlarged category $\mathcal{B}$, and one can consider condensation (in the extended string theory defined by $\mathcal{B}$) of boundary operators $f\in\mathrm{Hom}_{\mathcal{B}}(a,q)$. Suppose that we pick such a morphism $f_{aq}\in\mathrm{Hom}^1_{\mathcal{B}}(a,q)$ which defines a pseudocomplex $f=(f_{aq})$ in $\mathcal{B}$ over the objects a, q of $\mathcal{B}$, i.e. such that the tadpole cancellation constraint in $\mathcal{B}$ is satisfied (see figure 4):

$$Q^{\mathcal{B}}_{aq}f_{aq}=0 \Longleftrightarrow Q_{aa}f_{aq}+q_{aa}f_{aq}=0\,. \tag{5.4}$$

Condensation of f then produces a contracted category $\mathcal{B}_q$ which must be equivalent with a subcategory of $\mathcal{B}$ since $\mathcal{B}$ was assumed to be quasiunitary. In particular, $\mathcal{B}$ must contain an object $*_f$ such that $\mathrm{Hom}_{\mathcal{B}}(*_f,*_f)=\mathrm{Hom}_{\mathcal{B}_q}(*_f,*_f)$, i.e.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(*_f,*_f) &= \mathrm{Hom}_{\mathcal{B}}(a,a)\oplus\mathrm{Hom}_{\mathcal{B}}(q,q)\oplus\mathrm{Hom}_{\mathcal{B}}(a,q)\oplus\mathrm{Hom}_{\mathcal{B}}(q,a)\\
&= \mathrm{Hom}_{\mathcal{A}}(a,a)^{\oplus 4}\,,
\end{aligned}$$

where we used our assumptions (5.2). The BRST charge on this space is:

$$\begin{aligned}
Q^{\mathcal{B}}_{*_f,*_f}u = Q^{\mathcal{B}_q}_{*_f,*_f}u &= [Q^{\mathcal{B}}_{aa}u_{aa}-(-1)^{|u_{qa}|}u_{qa}f_{aq}]\oplus\\
&\oplus[Q^{\mathcal{B}}_{qq}u_{qq}+f_{aq}u_{qa}]\oplus[Q^{\mathcal{B}}_{aq}u_{aq}-(-1)^{|u_{qq}|}u_{qq}f_{aq}]\oplus[Q^{\mathcal{B}}_{qa}u_{qa}]
\end{aligned} \tag{5.5}$$

for $u=u_{aa}\oplus u_{qq}\oplus u_{aq}\oplus u_{qq}$ with $u_{ef}\in\mathrm{Hom}_{\mathcal{B}}(e,f)=\mathrm{Hom}_{\mathcal{A}}(a,a)$ for all $e,f\in\{a,q\}$. In view of our assumptions (5.3), this equals:

$$\begin{aligned}
Q^{\mathcal{B}}_{*_f,*_f}u &= [Q_{aa}u_{aa}-(-1)^{|u_{qa}|}u_{qa}f_{aq}]\oplus[Q_{aa}u_{qq}+q_{aa}u_{qq}-(-1)^{|u_{qq}|}u_{qq}q_{aa}+f_{aq}u_{qa}]\oplus\\
&\oplus[Q_{aa}u_{aq}+q_{aa}u_{aq}-(-1)^{|u_{qq}|}u_{qq}f_{aq}]\oplus[Q_{qa}u_{qa}-(-1)^{|u_{qa}|}u_{qa}q_{aa}]\,.
\end{aligned} \tag{5.6}$$

[10] The reason why this is only an assumption is that our description of D-brane condensation does not immediately tell us how to describe morphisms between pseudocomplexes whose underlying sets of objects are not disjoint. Assuming this form of morphisms and BRST operators amounts to the statement that any quasiunitary extension $\mathcal{B}$ of $\mathcal{A}$ contains the category of pseudocomplexes $p(\mathcal{A})$ alluded to above. This can be justified more formally but I prefer to avoid further technicalities.

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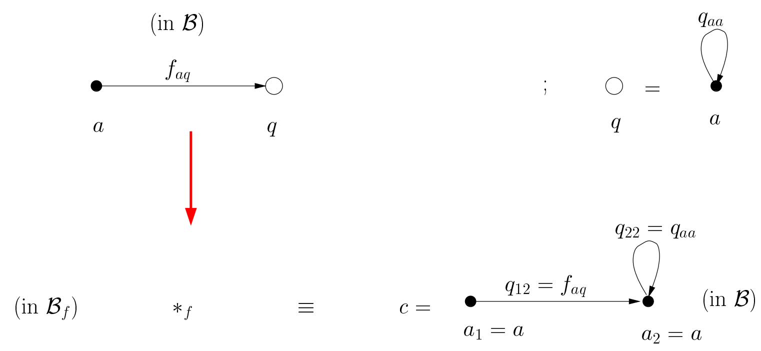

Figure 4: The pseudocomplex f over $\mathcal{B}$ and the associated generalized complex c over $\mathcal{A}$. In the upper left corner, we show f as a pseudocomplex over $\mathcal{B}$. The associated vacuum shift (condensation of the boundary operator f_{aq} between the objects a and q of $\mathcal{B}$) produces an object $*_f$ in the collapsed category $\mathcal{B}_f$, which is identified with the generalized complex c over $\mathcal{A}$. The latter can be viewed as an element of $\mathcal{B}$, if $\mathcal{B}$ is pseudounitary extension of $\mathcal{A}$. This type of construction can be used to argue that a quasiunitary extension $\mathcal{B}$ of $\mathcal{A}$ contains all generalized complexes over $\mathcal{A}$. The introduction of generalized complexes amounts intuitively to allowing for the formation of composites between a D-brane and its deformations.

It is clear that the object $*_f$ cannot be identified with a pseudocomplex s over $\mathcal{A}$, since the morphism space $\mathrm{Hom}(s,s)$ for a pseudocomplex cannot contain a direct power of the form $\mathrm{Hom}(a,a)^{\oplus 4}$ (this is due to the fact that pseudocomplexes are based on *sets* of objects of $\mathcal{A}$, and thus cannot contain repetitions of the same object). Hence a minimal quasiunitary extension $\mathcal{B}$ must consist of more than pseudocomplexes over $\mathcal{A}$. In fact, the object $*_f$ can be represented by a so-called generalized complex, namely by a *sequence* of objects and degree one morphisms of $\mathcal{A}$ satisfying a generalization of the tadpole cancellation constraint (a formal definition of generalized complexes is given below). For this, we define:

$$\begin{aligned} a_1 &= a_2 = a\,, \qquad q_{11} = q_{21} = 0\,, \\ q_{12} &= f_{aq}\,, \qquad q_{22} = q_{aa}\,, \end{aligned} \tag{5.7}$$

and view this data as a two-term sequence (a_1, a_2) of objects of $\mathcal{A}$ together with the degree one morphisms $q_{ij}(i,j = 1,2)$. It is easy to see that the tadpole cancellation constraints (5.1) and (5.4) are equivalent with:

$$Q_{a_i a_j} q_{ij} + \sum_k q_{kj} q_{ik} = 0\,. \tag{5.8}$$

Hence $c := [(a_i)_{i=1,2}, (q_{ij})_{i,j=1,2}]$ is a degree one generalized complex over $\mathcal{A}$ in the sense defined below (this is depicted in figure 4). It is shown below that generalized complexes form a dG category $c(\mathcal{A})$, in which the endomorphism space of c is:

$$\mathrm{Hom}_{c(\mathcal{A})}(c,c) = \oplus_{i,j=1}^{2} \mathrm{Hom}_{\mathcal{A}}(a_i, a_j) = \mathrm{Hom}(a,a)^{\oplus 4}\,, \tag{5.9}$$

with the BRST charge:

$$Q_{cc}u = \oplus_{i,j=1}^{2}\left[Q_{a_i,a_j}u_{ij} + \sum_k q_{kj}u_{ik} - \sum_k (-1)^{|u_{kj}|}u_{kj}q_{ik}\right] \tag{5.10}$$

for $u = \oplus_{i,j=1,2}u_{ij}$ with $u_{ij} \in \mathrm{Hom}(a_i, a_j) = \mathrm{Hom}(a, a)$. It is easy to see that this coincides with (5.6).

This simple example explains the need for introducing generalized complexes, and can be extended to show that in a certain sense the category of generalized complexes $c(\mathcal{A})$ constructed below gives the *minimal* quasiunitary extension of the string field theory based on $\mathcal{A}$.

5.2 Formal definition

We now proceed to give a formal definition of generalized complexes and of the category $c(\mathcal{A})$.

Definition 3. *A (degree one)* generalized complex *over $\mathcal{A}$ is given by the following data:*

(1) a sequence *$(a_i)_{i\in I}$ of objects of $\mathcal{A}$ (note that the objects a_i* need not be distinct*), where I is some finite set of indices.*

(2) for each $i, j \in I$, a morphism $q_{ij} \in \mathrm{Hom}(a_i, a_j)$. These morphisms are subject to the constraints:

$$Qq_{ij} + \sum_{k\in I} q_{kj}q_{ik} = 0\,. \tag{5.11}$$

We say that a generalized complex has degree k if $q_{ij} \in \mathrm{Hom}^k(a_i, a_j)$ for all $i, j \in I$.

Definition 4. *Given a dG category $\mathcal{A}$, its* quasiunitary cover *is the dG category $c(\mathcal{A})$ constructed as follows.*

(1) The objects of $c(\mathcal{A})$ are generalized complexes of degree one.

(2) If $q = [(a_i)_{i\in I}, (q_{ij})_{i,j\in I}]$ and $q' = [(a'_i)_{i\in J}, (q'_{ij})_{i,j\in J}]$ are two generalized complexes of degree one, then the space of morphisms $\mathrm{Hom}_{c(\mathcal{A})}(q, q')$ is given by:

$$\mathrm{Hom}(q, q') = \oplus_{i\in I, j\in J}\mathrm{Hom}(a_i, a'_j)\,, \tag{5.12}$$

(this is inspired by relation (4.5) and our previous example).

(3) *Given a third degree one generalized complex* $q'' = [(a''_i)_{i\in K}, (q''_{ij})_{i,j\in K}]$, *the composition of morphisms* $\mathrm{Hom}(q', q'') \times \mathrm{Hom}(q, q') \to \mathrm{Hom}(q, q'')$ *is given by:*

$$uv = \oplus_{i\in I,k\in K} \left[\sum_{j\in J} u_{jk} v_{ij}\right], \tag{5.13}$$

where $v = \oplus_{i\in I,j\in J} v_{ij} \in \mathrm{Hom}(q, q'), u = \oplus_{j\in J,k\in K} u_{jk} \in \mathrm{Hom}(q', q'')$, *with* $v_{ij} \in \mathrm{Hom}(a_i, a'_j)$ *and* $u_{jk} \in \mathrm{Hom}(a'_j, a''_k)$.

(4) *The BRST operator* $Q_{qq'} : \mathrm{Hom}(q, q') \to \mathrm{Hom}(q, q')$ *is defined via the analogue of (4.6):*

$$Q_{qq'} u = \oplus_{i\in I,j\in J} \left[Q_{a_i a'_j} u_{ij} + \sum_{k\in J} q'_{kj} u_{ik} - \sum_{l\in I} (-1)^{|u_{lj}|} u_{lj} q_{il}\right] \tag{5.14}$$

for $u = \oplus_{i\in I,j\in J} u_{ij} \in \mathrm{Hom}(q, q')$, *with* $u_{ij} \in \mathrm{Hom}(a_i, a'_j)$.

(5) *The grading on* $\mathrm{Hom}(q, q')$ *is induced from the grading on the spaces* $\mathrm{Hom}(a, b)$, *i.e.*

$$\mathrm{Hom}^k(q, q') = \oplus_{i\in I,j\in J} \mathrm{Hom}^k(a_i, a'_j)\,. \tag{5.15}$$

It is clear that a pseudocomplex $q = (q_{ab})_{a,b\in S}$ can be viewed as a (degree one) generalized complex upon choosing an enumeration $S = \{a_i | i \in I\}$ of its underlying set of objects (note that there are many such generalized complexes, differing by a choice in the enumeration of S). However, not every generalized complex is of this type, since for a generalized complex $q = [(a_i)_{i\in I}, (q_{ij})_{i,j\in I}]$ one can have repetitions in the sequence of objects $(a_i)_{i\in I}$. For example, all objects a_i could be identical, $a_i = a$ for all i. In this sense, a (degree one) generalized complex is a generalization of a pseudocomplex, and cannot be obtained in a simple fashion by performing a vacuum deformation of the original string field theory. This is, in particular, why passage to $c(\mathcal{A})$ gives something essentially new.

It is not hard to check that $c(\mathcal{A})$ is a dG category. The only slightly non-trivial statement is that the operators $Q_{qq'}$ are nilpotent. This results from a computation which I shall leave as an exercise for the diligent reader. Moreover, we can endow $c(\mathcal{A})$ with bilinear forms $(.,.) : \mathrm{Hom}(q', q) \times \mathrm{Hom}(q, q') \to \mathbf{C}$, defined as follows:

$$(u, v) = \sum_{i\in I,j\in J} (u_{ji}, v_{ij}) \tag{5.16}$$

for $u = \oplus_{j\in J,i\in I} u_{ji}$, $v = \oplus_{i\in I,j\in J} v_{ij}$ with $u_{ji} \in \mathrm{Hom}(a'_j, a_i)$ and $v_{ij} \in \mathrm{Hom}(a_i, a'_j)$. It is easy to check that the category $c(\mathcal{A})$ endowed with these forms satisfies Axioms

(1) and (2) of section 2 and hence defines a string field theory, which we shall call the *quasiunitary completion* of the string field theory based on $\mathcal{A}$.[11]

5.3 Quasiunitarity of $c(\mathcal{A})$

We now proceed to show that $c(\mathcal{A})$ gives a quasiunitary string theory in the sense of the previous subsection. The argument given below is pretty straightforward, but slightly involved, and can be skipped at a first reading of the paper.

Proposition 5. *The string field theory $c(\mathcal{A})$ is quasi-unitary.*

Proof. Let f be a degree one pseudocomplex over $c(\mathcal{A})$. We have to show that the contracted category $\mathcal{B} = c(\mathcal{A})_f$ is dG-equivalent with a subcategory of $c(\mathcal{A})$ (see appendix A for a precise definition of the notion of dG-equivalence). For simplicity, we assume that f is connected (it is easy to see that it suffices to check the statement for connected complexes.[12] We start by describing the contracted category $\mathcal{B}$.

Description of $\mathcal{B}$. Since f is a degree one pseudocomplex over $c(\mathcal{A})$, it is in particular a degree one generalized complex over $c(\mathcal{A})$, provided that we chose an enumeration of its set of objects, and we assume that such an enumeration has been chosen. Then $f = [(q^{(\alpha)})_{\alpha\in A}, (f^{(\alpha\beta)})_{\alpha\beta\in A}]$ for some set A, with $q^{(\alpha)}$ some degree one generalized complexes over $\mathcal{A}$ (i.e. objects of $c(\mathcal{A})$) and $f^{(\alpha\beta)} \in \mathrm{Hom}^1_{c(\mathcal{A})}(q^{(\alpha)}, q^{(\beta)})$, satisfying the tadpole cancellation constraint for $c(\mathcal{A})$:

$$Q^{(\alpha\beta)} f^{(\alpha\beta)} + \sum_{\gamma\in A} f^{(\gamma\beta)} f^{(\alpha\gamma)} = 0\,. \tag{5.17}$$

Here $Q^{(\alpha\beta)}$ is the BRST operator on $\mathrm{Hom}_{c(\mathcal{A})}(q^{(\alpha)}, q^{(\beta)})$.

Now, each $q^{(\alpha)} = [(a_i^{(\alpha)})_{i\in I_\alpha}, (q_{ij}^{(\alpha)})_{i,j\in I_\alpha}]$ is a degree one generalized complex over $\mathcal{A}$, with $(a_i^{(\alpha)})_{i\in I_\alpha}$ a family of objects of $\mathcal{A}$ and $q_{ij}^{(\alpha)} \in \mathrm{Hom}^1_{\mathcal{A}}(a_i^{(\alpha)}, a_j^{(\alpha)})$. Moreover, each f is a family of morphisms in the category $\mathcal{A}$:

$$f^{(\alpha\beta)} = \oplus_{i\in I_\alpha, j\in I_\beta} f_{ij}^{(\alpha\beta)} \text{with } f_{ij}^{(\alpha\beta)} \in \mathrm{Hom}^1_{\mathcal{A}}\left(a_i^{(\alpha)}, a_j^{(\beta)}\right), \tag{5.18}$$

and we have:

$$Q^{(\alpha\beta)} f^{(\alpha\beta)} = \oplus_{i\in I_\alpha, j\in I_\beta} \left[Q_{a_i^{(\alpha)} a_j^{(\beta)}} f_{ij}^{(\alpha\beta)} + \sum_{k\in I_\beta} q_{kj}^{(\beta)} f_{ik}^{(\alpha\beta)} + \sum_{k\in I_\alpha} f_{kj}^{(\alpha\beta)} q_{ik}^{(\alpha)} \right], \tag{5.19}$$

[11] It can be argued that this extension is minimal in the sense that any other extension contains it. This requires a slightly technical analysis which amounts to showing that any quasiunitary extension $\mathcal{B}$ of $\mathcal{A}$ must include all generalized degree one complexes over $\mathcal{A}$. This is a generalization of our the example we considered in subsection 5.1

[12] This follows from the observation that condensing two disjoint connected pseudocomplexes q, q' simultaneously (which is the same as condensing the disconnected pseudocomplex given by their union) is equivalent with first condensing q and then condensing q' in the category contracted along q.One can alternately extend the computations below to the case of disconnected pseudocomplexes, at the price of doubling the length of some formulas.

(where we used the fact that all components of f and q have degree one) and:

$$f^{(\beta\gamma)}f^{(\alpha\beta)} = \oplus_{i\in I_\alpha, j\in I_\beta}\left[\sum_{k\in I_\gamma} f^{(\beta\gamma)}_{kj} f^{(\alpha\beta)}_{ik}\right]. \tag{5.20}$$

Hence the tadpole cancellation condition (5.17) reduces to:

$$Q_{a_i^{(\alpha)},a_j^{(\beta)}} f^{(\alpha\beta)}_{ij} + \sum_{k\in I_\beta} q^{(\beta)}_{kj} f^{(\alpha\beta)}_{ik} + \sum_{k\in I_\alpha} f^{(\alpha\beta)}_{kj} q^{(\alpha)}_{ik} + \sum_{\gamma\in A, k\in I_\gamma} f^{(\gamma\beta)}_{kj} f^{(\alpha\gamma)}_{ik} = 0\,. \tag{5.21}$$

On the other hand, we have the tadpole cancellation constraints obeyed by each of the complexes $q^{(\alpha)}$:

$$Q_{a_i^{(\alpha)},a_j^{(\alpha)}} q^{(\alpha)}_{ij} + \sum_{k\in I_\alpha} q^{(\alpha)}_{kj} q^{(\alpha)}_{ik} = 0\,. \tag{5.22}$$

Consider now the contracted category $\mathcal{B} = c(\mathcal{A})_f$. Its objects are those objects of $c(\mathcal{A})$ which differ from all $q^{(\alpha)}$, plus a new objects $*_f$. Its morphism spaces are defined as above, and it sufficed to consider:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{B}}(f,f) &= \oplus_{\alpha,\beta\in A}\,\mathrm{Hom}_{c(\mathcal{A})}\left(q^{(\alpha)}, q^{(\beta)}\right)\\ &= \oplus_{\alpha,\beta\in A}\oplus_{i\in I_\alpha, j\in I_\beta}\,\mathrm{Hom}_{\mathcal{A}}\left(a_i^{(\alpha)}, a_j^{(\beta)}\right),\end{aligned} \tag{5.23}$$

and

$$\mathrm{Hom}_{\mathcal{B}}(q,f) = \oplus_{\alpha\in A}\mathrm{Hom}_{c(\mathcal{A})}(q, q^{(\alpha)}) = \oplus_{\alpha\in A}\oplus_{i\in J, j\in I_\alpha}\,\mathrm{Hom}_{\mathcal{A}}\left(b_i, a_j^{(\alpha)}\right), \tag{5.24}$$

where $q = [(b_i)_{i\in J}, (q_{ij})_{i,j\in J})]$ is an object of $c(\mathcal{A})$ which differs from all $q^{(\alpha)}$.

We shall need the BRST differentials on these spaces. On $\mathrm{Hom}_{\mathcal{B}}(f,f)$ we have the differential Q_{ff} given by:

$$Q_{ff}u = \oplus_{\alpha\beta\in A}\left[Q^{(\alpha\beta)}u^{(\alpha\beta)} + \sum_{\gamma\in A} f^{(\gamma\beta)}u^{(\alpha\gamma)} - \sum_{\gamma\in A}(-1)^{|u^{(\gamma\beta)}|}u^{(\gamma\beta)}f^{(\alpha\gamma)}\right], \tag{5.25}$$

where $u = \oplus_{\alpha\beta}u^{(\alpha\beta)} \in \mathrm{Hom}_{\mathcal{B}}(f,f)$, with $u^{(\alpha\beta)} = \oplus_{i\in I_\alpha, j\in I_\beta} u^{(\alpha\beta)}_{ij} \in \mathrm{Hom}_{c(\mathcal{A})}(q^{(\alpha)}, q^{(\beta)})$ and $u^{(\alpha\beta)}_{ij} \in \mathrm{Hom}_{\mathcal{A}}(a_i^{(\alpha)}, a_j^{(\beta)})$. This has the expansion:

$$\begin{aligned}Q_{ff}u = \oplus_{\alpha\beta\in A}\oplus_{i\in I_\alpha, j\in I_\beta}\Bigg[&Q_{a_i^{(\alpha)}a_j^{(\beta)}}u^{(\alpha\beta)}_{ij} + \sum_{k\in I_\beta} q^{(\beta)}_{kj}u^{(\alpha\beta)}_{ik} - \sum_{l\in I_\alpha}(-1)^{|u^{(\alpha\beta)}_{lk}|}u^{(\alpha\beta)}_{lk}q^{(\alpha)}_{il} + \\ &+ \sum_{\gamma\in A, k\in I_\gamma} f^{(\gamma\beta)}_{kj}u^{(\alpha\gamma)}_{ik} - \sum_{\gamma\in A, k\in I_\gamma}(-1)^{|u^{(\gamma\beta)}_{kj}|}u^{(\beta)}_{kj}f^{(\alpha\gamma)}_{ik}\Bigg].\end{aligned} \tag{5.26}$$

On $\mathrm{Hom}_{\mathcal{B}}(q,f)$, we have the operator:

$$Q_{qf}u = \oplus_{\alpha\in A}\left[Q_{qq^{(\alpha)}}u^{(\alpha)} + \sum_{\beta\in A} f^{(\beta\alpha)}u^{(\beta)}\right], \tag{5.27}$$

for $u = \oplus_{\alpha\in A}u^{(\alpha)}$ with $u^{(\alpha)} \in \mathrm{Hom}_{c(\mathcal{A})}(q, q^{(\alpha)})$. Upon expanding $u^{(\alpha)} = \oplus_{i\in J, j\in I_\alpha} u^{(\alpha)}_{ij}$, with $u^{(\alpha)}_{ij} \in \mathrm{Hom}(b_i, a^{(\alpha)}_j)$, we have:

$$Q_{qq^{(\alpha)}}u^{(\alpha)} = \oplus_{i\in J, j\in I_\alpha}\left[Q_{b_i, a^{(\alpha)}_j}u^{(\alpha)}_{ij} + \sum_{k\in I_\alpha} q^{(\alpha)}_{kj}u^{(\alpha)}_{ik} - \sum_{k\in J}(-1)^{|u^{(\alpha)}_{kj}|}u^{(\alpha)}_{kj}q_{ik}\right], \tag{5.28}$$

and

$$f^{(\beta\alpha)}u^{(\beta)} = \oplus_{i\in J, j\in I_\alpha}\left[\sum_{k\in I_\beta} f^{(\beta\alpha)}_{kj}u^{(\beta)}_{ik}\right]. \tag{5.29}$$

Hence we can expand:

$$\begin{aligned} Q_{qf}u = \oplus_{\alpha\in A}\oplus_{i\in J, j\in I_\alpha}&\left[Q_{b_i a^{(\alpha)}_j}u^{(\alpha)}_{ij} + \sum_{k\in I_\alpha} q^{(\alpha)}_{kj}u^{(\alpha)}_{ik} - \sum_{k\in J}(-1)^{|u^{(\alpha)}_{kj}|}u^{(\alpha)}_{kj}q_{ik} + \right.\\ &\left. + \sum_{\beta\in A, k\in I_\beta} f^{(\beta\alpha)}_{kj}u^{(\beta)}_{ik}\right]. \end{aligned} \tag{5.30}$$

The object of $c(\mathcal{A})$ associated with f. We now construct a degree one generalized complex $\mathbf{q}$ over $\mathcal{A}$ which can be identified with the generalized complex of generalized complexes given by f. We let $I = \sqcup_{\alpha\in A} I_\alpha$ be the disjoint union of the index sets I_α. Remember that I can be identified with the set of all pairs (α, i) with $\alpha \in A$ and $i \in I_\alpha$. We consider the family of objects of $\mathcal{A}$ indexed by I and defined through $a(\alpha, i) = a^{(\alpha)}_i$ for all $\alpha \in A$ and $i \in I_\alpha$. Next, we consider the morphisms $\mathbf{q}_{\alpha i,\beta j} \in \mathrm{Hom}^1(a_{\alpha i}, a_{\beta j}) = \mathrm{Hom}^1(a^{(\alpha)}_i, a^{(\beta)}_j)$ given by:

$$\begin{aligned} \mathbf{q}_{\alpha i,\alpha j} &= f^{(\alpha\alpha)}_{ij} + q^{(\alpha)}_{ij}\,, \\ \mathbf{q}_{\alpha i \beta j} &= f^{(\alpha\beta)}_{ij} \text{for } \alpha \neq \beta\,. \end{aligned} \tag{5.31}$$

It is not hard to see that $\mathbf{q}$ satisfies the tadpole cancellation constraint of $\mathcal{A}$, namely:

$$Q_{a^{(\alpha)}_i, a^{(\beta)}_j}\mathbf{q}_{\alpha i,\beta j} + \sum_{\gamma\in A, k\in I_\gamma}\mathbf{q}_{\gamma k \beta j}\mathbf{q}_{\alpha i \gamma k} = 0\,. \tag{5.32}$$

This follows by using the tadpole cancellation constraints (5.21) for the generalized complex f and (5.22) for the generalized complexes $q^{(\alpha)}$ and is left as an exercise for the reader.

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The next step is to compute the Hom spaces of $\mathbf{q}$ with itself and with with degree one generalized complexes $q = [(b_i)_{i\in J}, (q_{ij})_{i,j\in J}]$ of the type considered above. We have:

$$\mathrm{Hom}_{c(\mathcal{A})}(\mathbf{q},\mathbf{q}) = \oplus_{\alpha,\beta\in A}\oplus_{i\in I_\alpha, j\in I_\beta}\mathrm{Hom}_{\mathcal{A}}\left(a_i^{(\alpha)}, a_j^{(\alpha)}\right), \tag{5.33}$$

and

$$\mathrm{Hom}_{c(\mathcal{A})}(q,\mathbf{q}) = \oplus_{\alpha\in A}\oplus_{i\in J, j\in I_\alpha}\mathrm{Hom}_{\mathcal{A}}\left(b_i, a_j^{(\alpha)}\right), \tag{5.34}$$

which coincide with the spaces $\mathrm{Hom}_{\mathcal{B}}(f,f)$ and $\mathrm{Hom}_{\mathcal{B}}(q,f)$ computed above (see (5.23) and (5.24)). We shall now show that the BRST operators on these spaces also agree. Indeed, on the space $\mathrm{Hom}_{c(\mathcal{A})}(\mathbf{q},\mathbf{q})$ we have the operator $Q_{\mathbf{q},\mathbf{q}}$ given by:

$$Q_{\mathbf{q},\mathbf{q}}u = \oplus_{\alpha,\beta\in A}\oplus_{i\in I_\alpha, j\in I_\beta}\left[Q_{a_i^{(\alpha)},a_j^{(\beta)}}u_{ij}^{(\alpha\beta)} + \sum_{\gamma\in A, k\in I_\gamma}\mathbf{q}_{\gamma k,\beta j}u_{ik}^{(\alpha\gamma)} - \right.$$
$$\left. - \sum_{\gamma\in A, k\in I_\gamma}(-1)^{|u_{kj}^{(\gamma\beta)}|}u_{kj}^{(\gamma\beta)}\mathbf{q}_{\alpha i,\gamma k}\right],$$

where $u = \oplus_{\alpha,\beta\in A}\oplus_{i\in I_\alpha, j\in I_\beta}u_{ij}^{(\alpha\beta)}$, with $u_{ij}^{(\alpha\beta)}\in\mathrm{Hom}_{\mathcal{A}}(a_i^{(\alpha)}, a_j^{(\beta)})$. It is easy to see that this agrees with expression (5.26) for Q_{ff} upon substituting our definition (5.31) for $\mathbf{q}$.

On the space $\mathrm{Hom}_{c(\mathcal{A})}(q,\mathbf{q})$ we have the operator:

$$Q_{q\mathbf{q}}u = \oplus_{\alpha\in A}\oplus_{i\in J, j\in I_\alpha}\left[Q_{b_i,a_j^{(\alpha)}}u_{ij}^{(\alpha)} + \sum_{\beta\in A, k\in I_\beta}\mathbf{q}_{\beta k,\alpha j}u_{ik}^{(\beta)} - \right.$$
$$\left. - \sum_{k\in J}(-1)^{|u_{kj}^{(\alpha)}|}u_{kj}^{(\alpha)}q_{ik}\right], \tag{5.35}$$

where $u = \oplus_{\alpha\in A}\oplus_{i\in J, j\in I_\alpha}u_{ij}^{(\alpha)}$ with $u_{ij}^{(\alpha)}\in\mathrm{Hom}_{\mathcal{A}}(b_i, a_j^{(\alpha)})$. This agrees with (5.30) upon substituting the definition (5.31) of $\mathbf{q}$.

A similar computation can be done for the morphism spaces $\mathrm{Hom}_{\mathcal{B}}(f,q)$ and $\mathrm{Hom}_{c(\mathcal{A})}(\mathbf{q},q)$. This is almost identical with our computation of $\mathrm{Hom}_{\mathcal{B}}(q,f)$ and $\mathrm{Hom}_{c(\mathcal{A})}(q,\mathbf{q})$ above and leads once again to agreement.

Equivalence of $\mathcal{B} = c(\mathcal{A})_f$ with a subcategory of $c(\mathcal{A})$. It is now obvious how to define the desired dG-equivalence. One considers the full subcategory $\mathcal{C}(f)$ of $c(\mathcal{A})$ consisting of $\mathbf{q}$ and those degree one generalized complexes q over $\mathcal{A}$ which are distinct from the components $q^{(\alpha)}$ of f. One maps each such complex to the associated object of $\mathcal{B}$ and the complex $\mathbf{q}$ to the object f of $\mathcal{B}$. Moreover, one maps morphisms of $\mathcal{C}(f)$ into morphisms of $\mathcal{B}$ in the obvious manner. The previous computations show that the resulting functor F is a dG-functor, which is clearly a dG-equivalence. Hence

$\mathcal{B} = c(\mathcal{A})_f$ is dG-equivalent with the full subcategory $\mathcal{C}(f)$ of $c(\mathcal{A})$. Note that this equivalence depends on our initial choice of enumeration for the objects of f (which allowed us to view f as a generalized complex), and we obtain such an equivalence for every choice of enumeration. It is also easy to see that F preserves the corresponding bilinear forms. Since this holds for any connected pseudocomplex f over $c(\mathcal{A})$, we conclude that $c(\mathcal{A})$ is quasiunitary.

6. Conclusions

We discussed the general framework of associative open string theory in the presence of D-branes, relating it to the mathematical theory of differential graded categories. We showed that D-brane composite formation can be described systematically in this framework, through the general mechanism of condensation of boundary condition changing operators, and showed how this description can be extracted through the standard procedure of shifting the string vacuum.

We formulated a quasiunitarity constraint which encodes the condition that a string field theory in the presence of D-branes provides a self-consistent description of D-brane dynamics, in the sense that it is closed under formation of D-brane composites. We showed that this constraint can be satisfied by considering an enlargement $c(\mathcal{A})$ of a given D-brane category $\mathcal{A}$, its *quasiunitary cover*, which can be constructed as a category of generalized complexes over $\mathcal{A}$. Our results provide a general description of D-brane composite formation and represent a first step toward a better understanding of the structure of open string field theory in the presence of D-branes. This analysis is extremely general and can be applied to any associative string field theory.

It is clear that such a structural analysis could form a good foundation for gaining a better understanding of various dynamical issues in D-brane physics. Perhaps the most immediate application concerns a better understanding of K-theory as a classification of D-brane charge. Indeed, it seems likely that the ultimate formulation of D-brane charge should be a version of K-theory for a differential graded category (this should be a generalization of Quillen's K-theory of exact categories [9, 10]).[13] Since our approach gives a systematic treatment of D-brane composite formation (thus including the essential dynamics behind the original arguments for the relevance of K-theory in D-brane physics [5]) this offers the hope of a string-theoretic proof that a certain version of K-theory is indeed conserved by D-brane dynamics. Note that such an approach to K-theory charges would be extremely general, and in particular

[13] Quillen's categorical Q-construction gives the most general formulation of K-groups currently in wide use in mathematics. This requires an exact category, i.e. essentially a category with exact sequences. Exact categories seems unnatural from the point of view of this paper, since they are not produced in any direct fashion by the fundamental structure of open string theory with D-branes (though applications to topological open B-models do produce such objects).

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directly applicable to bosonic open string theory, for which the current understanding of K-theoretic charges is rather indirect [32]. Also note that, since such an approach would be based on the underlying category of the string field theory, it would take into account the effect of all massive string modes. This seems to be related to similar ideas proposed in [6] and would represent an off-shell counterpart to the approach of [31].

On a more speculative note, let me mention that our procedure (namely inclusion of all D-brane composites in order to obtain a unitary description of D-brane dynamics) is in a certain sense related to "second quantization" of D-branes (enlarging the state space such as to allow for D-brane generation/annihilation processes).[14] It is likely that there exists a connection between our procedure and M-theory, maybe through recent attempts to formulate the topological sector of M-theory as a theory of a topological membrane [17].

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A. Differential graded categories

This appendix collects some basic facts on differential graded categories. Good sources for more information are [2, 12]. We assume that the reader is familiar with the basic concepts of category theory, at the level of the first chapters in [1].

Definition 6. *A* graded category *is a category $\mathcal{A}$ whose morphism spaces are endowed with the structure of graded complex vector spaces, and such that all morphism compositions are bilinear maps of degree zero:*

$$|uv| = |u| + |v| \,, \qquad \textit{for } u \in \mathrm{Hom}(b,c) \,, \qquad v \in \mathrm{Hom}(a,b) \,, \tag{A.1}$$

where $|.|$ denotes the degree on morphisms.

Definition 7. *A* differential graded *(dG) category is a graded category $\mathcal{A}$ whose morphism spaces* $\mathrm{Hom}(a,b)$ *are graded differential complexes. Moreover, the linear applications* $\mathrm{Hom}(b,c) \otimes \mathrm{Hom}(a,b) \to \mathrm{Hom}(a,c)$ *induced by morphisms compositions*

[14] I thank C. Hofman for pointing this out.

are degree zero morphisms of graded complexes, where $\mathrm{Hom}(b,c)\otimes\mathrm{Hom}(a,b)$ *carries the structure of total complex:*

$$d(u\otimes v) = (du)\otimes v + (-1)^{|u|}u\otimes dv\,, \tag{A.2}$$

for $(u,v)\in\mathrm{Hom}(b,c)\times\mathrm{Hom}(a,b)$.

Note that the differentials d *on* $\mathrm{Hom}(a,b)$ *are taken to have degree* $+1$. *The condition on morphism compositions reads:*

$$d(uv) = (du)v + (-1)^{|u|}udv\,. \tag{A.3}$$

That is, d *acts as a derivation of morphism composition.*

One defines the notions of graded subcategory and dG subcategory in the obvious manner.

Definition 8. *Given a dG category* $\mathcal{A}$*, its* cohomology category *is the graded category* $H(\mathcal{A})$ *on the same objects as* $\mathcal{A}$*, and whose morphism spaces are given by the cohomology of the complexes* $\mathrm{Hom}_{\mathcal{A}}(a,b)$*:*

$$\mathrm{Hom}_{H(\mathcal{A})}(a,b) = H^*_d(\mathrm{Hom}_{\mathcal{A}}(a,b))\,. \tag{A.4}$$

The morphism compositions of $H(\mathcal{A})$ *are induced by those of* $\mathcal{A}$ *(the induced compositions are well defined due to condition (A.3)).*

Definition 9. *A (covariant) functor* $F:\mathcal{A}\to\mathcal{B}$ *between graded categories is a* graded functor *if the maps* $F:\mathrm{Hom}(a,b)\to\mathrm{Hom}(F(a),F(b))$ *are homogeneous of degree zero for any two objects* a,b *of* $\mathcal{A}$. *It is a* graded equivalence *if it is graded and admits a graded inverse.*

Definition 10. *A functor* $F:\mathcal{A}\to\mathcal{B}$ *between two dG categories is a* dG functor *if:*

(1) It is a graded functor

(2) It commutes with the differentials on morphisms, i.e.

$$F(du) = dF(u)\,, \qquad \textit{for all objects } a,b \textit{ of } \mathcal{A}\,. \tag{A.5}$$

A dG functor F *descends to a well-defined graded functor* $H(F):H(\mathcal{A})\to H(\mathcal{B})$ *between the associated cohomology categories. Moreover, we have* $H(FG)=H(F)H(G)$*, i.e. taking cohomology is itself a functorial operation.*

Definition 11. *A dG functor* $F:\mathcal{A}\to\mathcal{B}$ *is a* dG equivalence *if it admits a dG-functor as an inverse.*

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